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ON SOME INTEGRAL EQUATIONS WITH LOCALLY FINITE MEASURES AND L^∞-PERTURBATIONS

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ABSTRACT

Let $g \in C(R)$, $f \in L^1_{loc}(R^+)$ and let μ be a real locally finite positive definite Borel measure on R^+. We investigate a relation between the solutions of the nonlinear scalar Volterra equation

$$x'(t) + \int_{[0,t]} g(x(t-s))d\mu(s) = f(t), \quad t \in R^+, \ x(0) = x_0 ,$$

and the solutions of linear equations with the same data

$$z_\lambda'(t) + \lambda \int_{[0,t]} z_\lambda(t-s)d\mu(s) = f(t), \quad t \in R^+, \quad z_\lambda(0) = x_0 , \quad \lambda > 0 .$$

This relation, when combined with results (established in this paper) on the global size of solutions of certain limit equations

$$y(t) + \int_{R^+} g(y(t-s))a(s)ds = 0 , \quad t \in R ,$$

allows us to obtain new asymptotic results for the solution $x(t)$ in the case when both μ and f are large in a precise sense.

AMS(MOS) Subject Classification: 45D05, 45M05, 45G10

Key words: Volterra equations, nonlinear integral equations, asymptotic behavior, frequency domain methods

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SIGNIFICANCE AND EXPLANATION

In the construction of mathematical models of technical and physical systems one is frequently led to equations in which the current rate of change $(= \frac{dx}{dt})$ of the state of the system $(= x(t)$ at time $t)$ is a function not only of $x(t)$, but also of $x(\tau)$ for past times $\tau < t$. Specifically, one obtains Volterra integrodifferential equations, exemplified by

$$\text{(E)} \qquad \frac{dx}{dt} + \int_0^t g(x(t-s))d\mu(s) = f(t), \quad x(0) = x_0, \quad t > 0.$$

Here $f(t)$ is the external input, $\mu(t)$ is the feedback kernel, $g(x)$ is in general a nonlinear function of x. By letting $\mu(t)$ have discontinuities we realize that (E) includes a large class of differential-delay equations.

The key problem concerning (E) is the behavior of $x(t)$ for large values of t. In particular one is interested in whether the solutions $x(t)$ remain bounded and in case they do, whether $x(t)$ tends to a limit when $t \to \infty$, or whether the system continues to oscillate. The present report analyzes these questions and continues work begun in MRC Technical Summary Report 2152. We are in particular interested in the case when the variation of the feedback kernel is large, in the sense that $\mu(t)$ is not of bounded variation over the positive half-axis. Such kernels are frequent in applications; let for example $d\mu(s) = b(s)ds$ with $b(s) = (\cos s)s^{-\alpha}$, $0 < \alpha < 1$. The second key feature of this report is that we do allow large input functions $f(t)$ in that we only assume $f(t) \to 0$ as $t \to \infty$.

Our main statement (Theorem 4) essentially follows from two auxiliary results, both of which are established in this report. The first is a relation between the solutions of (E) and the solutions z_λ of the linear equations with the same data

$$\text{(L)} \qquad \frac{dz_\lambda}{dt} + \lambda \int_0^t z_\lambda(t-s)d\mu(s) = f(t), \quad t > 0, \quad z_\lambda(0) = x_0,$$

where λ is a positive parameter. The second concerns conditions under which the solutions $y(t)$ of certain limit equations

$$y(t) + \int_0^\infty g(y(t-s))a(s)ds = 0, \quad t \in R,$$

satisfy $\int_{-\infty}^\infty |y'(\tau)|^2 d\tau < \infty$, $\int_{-\infty}^\infty |g(y(\tau))|^2 d\tau < \infty$.

The results of the report are formulated in Theorems 1-5. Theorems 1 and 4 give conditions under which bounded solutions of (E) decay to zero as $t \to \infty$. Theorems 2 and 3 constitute auxiliary results paving the way for Theorem 4; but as they are of independent interest we prefer to state them separately. Theorem 5 gives conditions under which the solutions of (E) remain bounded.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

ON SOME INTEGRAL EQUATIONS WITH LOCALLY FINITE MEASURES AND L^∞-PERTURBATIONS

Stig-Olof Londen

1. INTRODUCTION

In this paper we demonstrate a certain connection between the asymptotic behavior of the solutions of the nonlinear scalar Volterra equation

$$x'(t) + \int_{[0,t]} g(x(t-s))d\mu(s) = f(t), \quad t \in R^+ = [0,\infty), \quad x(0) = x_0 , \tag{1.1}$$

and the corresponding behavior of the solutions of the linear equations with the same data,

$$z_\lambda'(t) + \lambda \int_{[0,t]} z_\lambda(t-s)d\mu(s) = f(t), \quad t \in R^+, \; z_\lambda(0) = x_0 , \quad \lambda > 0. \tag{1.2}$$

As a consequence of this connection we obtain some new asymptotic results on (1.1) in the case when both μ and f are large.

In the equations above g,μ,f,x_0 are given, λ is a positive parameter, while x, z_λ stand for the solutions. These solutions are always assumed to exist for $t \in R^+$, to be locally bounded, and to satisfy the corresponding equations a.e. on R^+. Throughout the article the following basic hypotheses on g,μ,f will be made:

(1.3)
(i) $g \in C(R)$,
(ii) μ is a real, locally finite, positive definite measure on R^+ ,
(iii) $f \in L^1_{loc}(R^+)$.

Define $Q(\varphi,\mu,T)$ for $\varphi \in L^2_{loc}(R^+)$, $T > 0$, by

$$Q(\varphi,\mu,T) = \int_0^T \varphi(t)(\varphi*\mu)(t)dt, \quad \text{where} \quad (\varphi*\mu)(t) \overset{\text{def}}{=} \int_{[0,t]} \varphi(t-s)d\mu(s) , \tag{1.4}$$

and let $x \in L^\infty(R^+)$. Then, as is well-known [8,9], a large amount of information concerning the asymptotic behavior of $x(t)$ can be obtained provided one succeeds in establishing

$$\sup_{T>0} Q(x,\mu,T) < \infty . \tag{1.5}$$

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Note that if in addition to $x \in L^\infty(R^+)$ one takes f small, i.e. $f \in L^1(R^+)$, then (1.5) immediately follows.

If f merely satisfies

(1.6) $$f \in L^\infty_{loc}(R^+) , \qquad \lim_{t\to\infty} f(t) = 0 ,$$

then the asymptotic analysis of $x(t)$ becomes significantly more difficult as (1.5) is now out of reach. However, by taking μ small enough, in particular by assuming

(1.7) $$\int_{R^+} t d|\mu|(t) < \infty ,$$

and by working with the limit equation *corresponding* to (1.1)

(1.8) $$y'(t) + \int_{R^+} g(y(t-s))d\mu(s) = 0 , \qquad t \in R ,$$

one may even now obtain asymptotic results on bounded solutions of (1.1), [3,11]. Observe furthermore that if in addition to (1.3i) $g(x)$ is taken locally Lipschitzian then (1.7) may be weakened to μ finite, [4].

The aim of the present work was originally to extend the results of [3,4,11] so as to apply to equations with μ only locally finite without excluding the possibility that f satisfies only (1.6). However, making use of a simple device we have in fact been able to connect the asymptotics of (1.1) and (1.2) and thus to reduce (under certain hypotheses) the asymptotic analysis of (1.1) to that of (1.2). The fact that (1.2) can be explicitly solved for z_λ, independently of the size of μ and f, then allows us to realize our original goal. An analogous approach, which however uses the integral resolvent, has been applied in [2, Theorem 3] to obtain a result on the integrated version of (1.1).

Our main results are Theorems 1 and 4. The former has the advantage of having a short and lucid proof. Also observe the important point that nothing but continuity is imposed on g. The assumptions of Theorem 1 do however include a moment condition, (1.11), on the second derivative of the differential resolvent of μ. Although this condition is satisfied (Lemma 1 below) for $d\mu(t) = a(t)dt$ with $a(t)$ nonintegrable but sufficiently monotone it is still the case that verification of (1.11) in general is quite hard if μ is only locally finite. It should also be observed that Theorem 1 requires $\hat{\mu}(\omega)$ to be finite for $\omega \neq 0$, thus excluding cases like $d\mu = a(t)dt$ with $a(t) = t^{-1/2}\cos t$.

One is consequently motivated to try to remove (1.11), (1.12). This is done in a series of steps, Theorem 2-4. Theorems 2 and 3 constitute auxiliary results but as they are of independent interest we prefer to state them separately. Observe that these statements concern equations with a finite measure α. Theorem 4 then corresponds to Theorem 1 but (1.11), (1.12) are now absent from the assumptions. Certain other conditions have instead been added, in particular on $g(x)$. These additional assumptions on g have the advantage of being easily checked and they are not overly restrictive. The added assumption that $g(x)$ be locally Lipschitzian is basic to the approach we use. The remaining additional hypotheses on g roughly speaking result from the fact that in Theorem 2 we establish $g(y(t)) \in L^2(R)$ - for which some condition of type (1.24) is needed if $0 \in Z$ - and not $[y(t) + g(y(t))] \in L^2(R)$ - which very likely only requires that g satisfies some smoothness condition. Although the latter conclusion undoubtedly is the natural one (under the assumptions on α made in Theorem 2) we have not been able to establish it without any sign condition on g.

Our last result, Theorem 5, states a new boundedness result on (1.1). It displays a connection between the existence of bounded solutions of (1.1) and the total variation of solutions of (1.2).

THEOREM 1. Let (1.3) hold and assume $r \in LAC(R^+)$ satisfies

(1.9) $$r'(t) + (r * \mu)(t) = 0 \quad \text{a.e. on } R^+, \quad r(0) = 1,$$

(1.10) $$r' \in (L^1 \cap NBV)(R^+),$$

(1.11) $$\int_{R^+} t\, d|r'|(t) < \infty .$$

Suppose

(1.12) $$|\hat{\mu}(\omega)| < \infty, \quad \omega \neq 0,$$

and let the set Z defined by $Z = \{\omega | \omega \neq 0, \ \mathrm{Re}\, \hat{\mu}(\omega) = 0\}$ be at most denumerable and such that

(1.13) $$\mathrm{Im}\, \hat{\mu}(\omega) = 0, \quad \omega \in Z.$$

Finally let x, z satisfy respectively (1.1) and (1.2) with $\lambda = 1$ and be such that

(1.14) $$x \in (LAC \cap L^\infty)(R^+), \quad z \in LAC(R^+).$$

Then, if

(1.15) $$\lim_{t\to\infty} z(t) = z(\infty)$$

exists (and is finite) one has

(1.16) $$\lim_{t\to\infty} [x(t+d) - x(t)] = 0 , \quad \forall\, d > 0 ,$$

(1.17) $$\lim_{t\to\infty} [r(\infty)x(t) + [1 - r(\infty)]g(x(t))] = z(\infty).$$

If in addition $\lim_{t\to\infty} z'(t) = 0$, then $\lim_{t\to\infty} x'(t) = 0$.

For an equivalent formulation of (1.17) see the conclusion of Theorem 4 and also Lemma 2 below.

By $\hat{\mu}(\omega)$, $\omega \neq 0$, we mean $\lim_{\substack{s\to i\omega \\ \mathrm{Re}\, s > 0}} \tilde{\mu}(s)$ where $\tilde{\mu}(s) = \int_{R^+} e^{-st} d\mu(t)$. To see that this is well-defined note at first that as μ is a positive definite measure then μ is a tempered distribution [8, p. 229] and so the Laplace transform $\tilde{\mu}(s)$ exists for $\mathrm{Re}\, s > 0$. Then observe that by (1.9)

$$-\int_{R^+} e^{-st} dr'(t) = s\tilde{\mu}(s)[s + \tilde{\mu}(s)]^{-1}, \qquad \mathrm{Re}\, s > 0 .$$

By (1.10) the left side is continuous for $\mathrm{Re}\, s \geq 0$. Hence

$\lim_{\substack{s\to i\omega \\ \mathrm{Re}\, s > 0}} s\tilde{\mu}(s)[s + \tilde{\mu}(s)]^{-1}$ exists for $\omega \in R$. One concludes that $\lim_{s\to i\omega, \mathrm{Re}\, s > 0} \tilde{\mu}(s)$ exists, possibly infinite, for $\omega \neq 0$. The assumption (1.12) does however exclude this last possibility.

Concerning (1.10) note that this condition is (locally with respect to ω) weaker than the assumption $r \in L^1(R^+)$. This is seen as follows. The Fourier transforms of r', dr' may be written respectively as

$$-\frac{1}{i\omega[\hat{\mu}]^{-1} + 1} , \qquad -\frac{i\omega}{i\omega[\hat{\mu}]^{-1} + 1} , \qquad \omega \neq 0.$$

Thus (1.10) requires (locally) $i\omega[\hat{\mu}]^{-1}$ to behave as the transform of an $L^1(R)$-function whereas the assumption $r \in L^1(R^+)$ imposes (locally) the same behavior on $[\hat{\mu}]^{-1}$. Not even the usual transform condition $s + \tilde{\mu}(s) \neq 0$, $\mathrm{Re}\, s \geq 0$, need hold. Thus if for

example $\hat{\mu} = \omega^2 + 2^{-1}i\omega$, $|\omega| < 1$ (with a sufficiently smooth extension of $\hat{\mu}$ to $|\omega| >$ 1) then the conditions of Theorem 1 on the differential resolvent $r(t)$ are satisfied.

In applications one of course frequently has $r(\infty) = 0$. In this case (1.17) reduces to $\lim_{t\to\infty} g(x(t)) = z(\infty)$.

A class of only locally finite positive definite measures for which the corresponding differential resolvents do satisfy (1.10), (1.11) is given by

LEMMA 1. Let $d\mu = a(t)dt$ where $a(t)$ is nonnegative, nonincreasing and convex on R^+ with $a \in L^1(0,1)$ and $s + \tilde{a}(s) \neq 0$, $\mathrm{Re}\, s > 0$. Then $r \in L^1(R^+)$ and (1.10) hold. If in addition $-a'(t)$ is convex then (1.11) is satisfied.

The fact that $r \in L^1(R^+)$ under the assumptions of Lemma 1 is proved in [7]. The assertions (1.10), (1.11) follow by straightforward estimates making use of [7, Lemma 1], [1, Lemma 5.1]. See [5] for details.

It should finally be observed that Theorem 1 extends earlier work [4] even if μ is finite. This is true because we only assume continuity on g.

Our next result constitutes a first step towards eliminating (1.11), (1.12) from the hypothesis of Theorem 1. It gives conditions under which the global size of the bounded solutions of the limit equation

$$y(t) + \int_{R^+} g(y(t-s))\alpha([0,s])ds = 0, \quad t \in R, \tag{1.18}$$

is sufficiently small, in case α is finite and $\alpha(R^+) = 0$.

Define $a(t) = \alpha([0,t])$, $\hat{\alpha}(\omega) = \int_{R^+} e^{-i\omega t}\,d\alpha(t)$, $\hat{a}(\omega) = \int_{R^+} e^{-i\omega t}a(t)dt$.

THEOREM 2. Let

(1.19) $g(x)$ be locally Lipschitzian, $x \in R$,

(1.20) α be a real, finite, positive definite Borel measure on R^+,

(1.21) $a \in L^1(R^+)$.

Define Z by $Z = \{\omega \mid \mathrm{Re}\,\hat{\alpha}(\omega) = 0\}$ and suppose that Z can be written as the union of three pairwise disjoint sets Z_1, Z_2, $\{0\}$, such that

$$\mathrm{Im}\,\hat{\alpha}(\omega) = 0, \quad \omega \in Z_1, \tag{1.22}$$

$$\mathrm{Im}\,\hat{\alpha}(\omega) \neq 0, \quad \omega \in Z_2; \quad \inf_{\omega \in Z_2 \cup \{0\}} \mathrm{Re}\,\hat{a}(\omega) > 0. \tag{1.23}$$

Finally assume that for some $K > 0$

$$(1.24)\qquad xg(x) \geq 0\ , \quad |x| \leq K\ .$$

Define $Y_K = \{y | y \in LAC(R),\ y$ *satisfies* (1.18), $\|y\|_{L^\infty(R)} \leq K\}$. *Then*

$$(1.25)\qquad \sup_{y \in Y_K} \|g(y(t))\|_{L^2(R)} < \infty\ , \quad \sup_{y \in Y_K} \|y'(t)\|_{L^2(R)} < \infty\ .$$

Observe that if y satisfies (1.18) then y also satisfies

$$(1.26)\qquad y'(t) + \int_{R^+} g(y(t-s))d\alpha(s) = 0 \quad \text{a.e.} \quad \text{on} \quad R\ .$$

Also note that (1.21) and the second part of (1.23) imply that Z_2 must be compact.

From Theorem 2 one may deduce the following result concerning the asymptotic behavior of solutions of

$$(1.27)\qquad x'(t) + \int_{[0,t]} g(x(t-s))d\alpha(s) = f(t), \quad t \in R^+\ , \quad x(0) = x_0\ .$$

THEOREM 3. *Let* g, α *be as in Theorem* 2 *and suppose* f *is such that*

$$(1.28)\qquad f \in L^\infty_{loc}(R^+), \quad \lim_{t\to\infty} f(t) = 0\ ,$$

$$(1.29)\qquad \lim_{t\to\infty} \int_0^t f(s)ds = F \quad \text{with} \quad |F| < \infty\ .$$

Let $x \in LAC(R^+)$ *be the solution of* (1.27) *and assume*

$$(1.30)\qquad \|x\|_{L^\infty(R^+)} < K\ ,$$

where K *is as in* (1.24). *Then*

$$(1.31)\qquad \lim_{t\to\infty} x'(t) = 0\ ,$$

$$(1.32)\qquad \lim_{t\to\infty} [x(t) + g(x(t)) \int_{R^+} a(s)ds] = F + x_0\ .$$

From the above one finally obtains an asymptotic result on the bounded solutions of (1.1) with μ assumed only locally finite and without (1.11), (1.12).

THEOREM 4. *Assume* (1.3) *and* (1.19) *hold. Also let*

$$(1.33)\qquad \operatorname{Im} \hat{\mu}(\omega) = 0 \quad \text{for} \quad \omega \in Z = \{\omega | \omega \neq 0,\ \operatorname{Re} \hat{\mu}(\omega) = 0\}\ ,$$

$$(1.34)\qquad xg(x) > 0\ , \quad x \neq 0\ ,$$

$$(1.35)\qquad \liminf_{|x|\to 0} x^{-1}g(x) > 0\ .$$

Suppose that there exists $\delta > 0$ *such that the solution* r_λ *of*

$$(1.36)\qquad r_\lambda'(t) + \lambda(r_\lambda * \mu)(t) = 0\ , \quad \text{a.e. on} \quad R^+, \quad r_\lambda(0) = 1\ ,$$

satisfies

(1.37) $$r'_\lambda \in L^1(R^+) , \qquad \lambda \in (0,\delta) ,$$

(1.38) $$r'_\lambda \in NBV(R^+) , \qquad \lambda \in (0,\delta) ,$$

and assume

(1.39) $$\lim_{\omega\to 0} i\omega[\hat{\mu}(\omega)]^{-1} \text{ is finite} .$$

Finally let $x \in (LAC \cap L^\infty)(R^+)$, $z_\lambda \in LAC(R^+)$ *satisfy* (1.1), (1.2) *respectively. Then, if for* $\lambda \in (0,\delta)$

(1.40) $$\lim_{t\to\infty} z'_\lambda(t) , \quad \lim_{t\to\infty} z_\lambda(t) \text{ both exist, are finite, with } z'_\lambda(\infty) = 0 ,$$

then

(1.41) $$\lim_{t\to\infty} x'(t) = 0 ,$$

(1.42) $$\lim_{t\to\infty} \{x(t) + \gamma^{-1}g(x(t))\} = x_0 + \lim_{s\to 0+, s \text{ real}} \tilde{f}(s) ,$$

provided $\gamma \overset{\text{def}}{=} \lim_{\omega\to 0} i\omega[\hat{\mu}(\omega)]^{-1} > 0$ *and*

(1.43) $$\lim_{t\to\infty} g(x(t)) = \lim_{\substack{s\to 0+\\ s \text{ real}}} \{s[\tilde{\mu}(s)]^{-1}\tilde{f}(s)\} ,$$

provided $\gamma = 0$.

Note that the existence of all the limits in (1.41)-(1.43) is part of the conclusion.

The comments made after Theorem 1 concerning the existence of $\hat{\mu}$ and the size of r are still valid; thus $r_\lambda \in L^1(R^+)$ need not necessarily hold. Also observe that none of the size assumptions on the derivatives of r_λ is supposed to be uniform with respect to λ .

The following Lemma 2 throws some light on the condition (1.39) and will be used in the proof of Theorem 4. It also shows that γ is by necessity nonnegative.

LEMMA 2. *Let* μ *be a locally finite positive definite Borel measure on* R^+ *and assume* (1.37) *holds. Then* $\lim_{\omega\to 0} i\omega[\hat{\mu}(\omega)]^{-1}$ *exists. Moreover, either the limit is finite, real or*

(1.44) $$\lim_{\omega\to 0} |i\omega[\hat{\mu}(\omega)]^{-1}| = \infty .$$

In case (1.44) holds one has $r_\lambda(\infty) = 1$, $\lambda \in (0,\delta)$. In case the limit is finite one has

$$(1.45) \qquad 0 \le \lim_{\omega \to 0} i\omega[\hat{\mu}(\omega)]^{-1} = \lambda r_\lambda(\infty)[1 - r_\lambda(\infty)]^{-1} < \infty ,$$

$$(1.46) \qquad 1 - r_\lambda(\infty) > 0 , \quad \lambda \in (0,\delta) ,$$

and consequently $0 \le r_\lambda(\infty) < 1$.

Conversely let (1.37), (1.46) be satisfied. Then $\lim_{\omega \to 0} i\omega[\hat{\mu}(\omega)]^{-1}$ exists and is finite, real and nonnegative.

Proof of Lemma 2. From (1.37), for $\lambda \in (0,\delta)$,

$$(1.47) \qquad 1 - r_\lambda(\infty) = \lim_{\omega \to 0} [-\hat{r}_\lambda(\omega)] = \lim_{\omega \to 0} \lambda\hat{\mu}(\omega)[i\omega + \lambda\hat{\mu}(\omega)]^{-1}.$$

Suppose at first that for some $\varepsilon > 0$, $\hat{\mu}(\omega) \neq 0$, $\omega \in (-\varepsilon,0) \cup (0,\varepsilon)$. Then

$$(1.48) \qquad 1 - r_\lambda(\infty) = \lim_{\omega \to 0} [i\omega[\lambda\hat{\mu}(\omega)]^{-1} + 1]^{-1} \text{ exists, is finite, real and } \ge 0 ,$$

for $\lambda \in (0,\delta)$. (The nonnegativity follows from the fact that by the positive definiteness of μ we have $|r_\lambda(\infty)| \le 1$. Also note that if $\hat{\mu}(\omega) = \infty$ for some $\omega \neq 0$ then we define $[\hat{\mu}(\omega)]^{-1} = 0$). Thus $\gamma \stackrel{\text{def}}{=} \lim_{\omega \to 0} i\omega[\hat{\mu}(\omega)]^{-1}$ exists and is either finite, real or (1.44) holds. In the latter case one has $r_\lambda(\infty) = 1$ by (1.48). If γ is finite, then again by (1.48) $0 < [\lim_{\omega \to 0} i\omega[\lambda\hat{\mu}(\omega)]^{-1} + 1]^{-1} \le 2$ and so

$$-2^{-1} \le \lambda^{-1}\gamma < \infty , \quad \lambda \in (0,\delta).$$

Suppose $\lambda_0^{-1}\gamma = \eta \in [-\tfrac{1}{2},0)$ for some $\lambda_0 \in (0,\delta)$. Then by choosing $\lambda_1 = -\lambda_0\eta$ (note that $\lambda_1 \in (0,\delta)$) we obtain $\lambda_1^{-1}\gamma = -1$ which is false. Thus the inequalities in (1.45) hold. The equality is easily established using (1.48).

In case there exists $\omega_n \to 0$ $(\omega_n \neq 0)$ such that $\hat{\mu}(\omega_n) = 0$ one immediately has $r_\lambda(\infty) = 1$ and $\lim_{|\omega| \to 0} |i\omega[\hat{\mu}(\omega)]^{-1}| = \infty$.

Conversely, suppose

$$0 < 1 - r_\lambda(\infty) = \lim_{\omega \to 0} \lambda\hat{\mu}(\omega)[i\omega + \lambda\hat{\mu}(\omega)]^{-1} < \infty , \qquad \lambda \in (0,\delta) .$$

Then

$$(1.49) \qquad 0 < \lim_{\omega \to 0} [i\omega[\lambda\hat{\mu}(\omega)]^{-1} + 1]^{-1} < \infty , \qquad \lambda \in (0,\delta)$$

and hence $\gamma \overset{\text{def}}{=} \lim_{\omega\to 0} i\omega[\hat{\mu}(\omega)]^{-1}$ exists and is finite, real. Suppose $\lambda_0^{-1}\gamma \in (-1,0)$ for some $\lambda_0 \in (0,\delta)$. (By (1.49) $\lambda^{-1}\gamma > -1$.) Then by choosing $\lambda_1 \in (0,\delta)$ small enough we obtain $\lambda_1^{-1}\gamma = -1$ which violates (1.49). The Lemma is proved.

Our last result concerns the existence of bounded solutions of (1.1).

THEOREM 5. Assume (1.3) holds and let

(1.50) $$|g(x)| \le c[1 + G(x)] \ ; \quad G(x) \ge \varepsilon x^2 - c \ ; \qquad x \in R \ ,$$

for some $c, \varepsilon > 0$, where $G(x) \overset{\text{def}}{=} \int_0^x g(u)du$. Let x, z_λ be locally absolutely continuous solutions of (1.1), (1.2) respectively and suppose that for some $\delta > 0$

(1.51) $$z_\lambda' \in L^1(R^+) \ , \qquad \lambda \in (0,\delta) \ .$$

Then

(1.52) $$\sup_{t\in R^+} |x(t)| < \infty \ .$$

Earlier boundedness results on (1.1), see [6,10] have required $f \in L^p(R^+)$ with $p = 1$ or 2. It is however easily checked that (1.51) translates into $f' \in L^1(R^+)$, provided $r_\lambda \in L^1(R^+)$, and thus Theorem 5 constitutes a significant generalization as compared to previous results.

2. PROOF OF THEOREM 1.

Convolve (1.1) with r and use (1.9). This gives

(2.1) $$r * x' - r' * g(x) = r * f .$$

Note that if both f_1, f_2 are measurable functions defined on R^+, then $f_1 * f_2 \overset{\text{def}}{=} \int_0^t f_1(t-s)f_2(s)ds$. An integration of the first term on the left side of (2.1) by parts results in

(2.2) $$x(t) - \int_0^t h(x(t-s))r'(s)ds = z(t) , \quad t \in R^+$$

where $h(x) \overset{\text{def}}{=} g(x) - x$, $x \in R$, and where we have used the fact that $z = x_0 r + r * f$. Differentiate (2.2) and define $c = -r'$. This yields

(2.3) $$x'(t) + \int_{[0,t]} h(x(t-s))dc(s) = z'(t), \quad t \in R^+.$$

From (1.9) follows after straightforward computations ($\hat{c} \overset{\text{def}}{=} \int_{R^+} e^{-i\omega t} dc(t)$; recall that $c(0) = c(\infty) = 0$, $c \in NBV(R^+)$),

(2.4) $$\text{Re } \hat{c}(\omega) = \omega^2 \text{Re } \hat{\mu}(\omega)|i\omega + \hat{\mu}(\omega)|^{-2} , \quad \omega \neq 0 ,$$

(2.5) $$\text{Im } \hat{c}(\omega) = \omega^2 \text{Im}\hat{\mu}(\omega)|i\omega + \hat{\mu}(\omega)|^{-2} + \omega|\hat{\mu}(\omega)|^2|i\omega + \hat{\mu}(\omega)|^{-2} , \quad \omega \neq 0 ,$$

(2.6) $$\hat{c}(0) = 0 .$$

As μ is positive definite we have $\text{Re } \hat{\mu} > 0$, $\omega \in R, \omega \neq 0$, and hence

(2.7) $$\text{Re } \hat{c}(\omega) \geq 0 , \quad \omega \in R ,$$

and by (1.12), (2.4)

(2.8) $$\text{Re } \hat{c}(\omega) = 0 , \quad \text{iff} \quad \omega \in Z \cup \{0\} .$$

But by (1.13), (2.6)

(2.9) $$\text{Im } \hat{c}(\omega) = 0 \quad \text{if} \quad \omega \in Z \cup \{0\} .$$

From (1.3), (1.10), (1.11), (1.14), (1.15), (2.7)-(2.9) it follows that we may apply [9, Corollary 3b] to the equation (2.2). This gives (1.16) and

(2.10) $$\lim_{t\to\infty} \{x(t) + h(x(t))[1 - r(\infty)]\} = z(\infty) .$$

Substitute the expression for $h(x)$ to get (1.17). Provided $\lim_{t\to\infty} z'(t) = 0$ we obtain $\lim_{t\to\infty} x'(t) = 0$ from [9, Theorem 1b].

3. PROOF OF THEOREM 2.

For $t > 0$ we define

$$(3.1) \qquad m_t^2 = \sup_{y \in Y_K} \int_{-t}^{t} |g(y(\tau))|^2 d\tau .$$

Assume $\lim_{t\to\infty} m_t^2 = \infty$, otherwise the first part of (1.25) holds. Then choose for each $t > 0$ $y_t \in Y_K$ such that

$$(3.2) \qquad \int_{-t}^{t} |g(y_t(\tau))|^2 d\tau = m_t^2 .$$

As Y_K is translation invariant one also has

$$(3.3) \qquad \sup_{\substack{y \in Y_K \\ s \in R}} \int_{s-t}^{s+t} |g(y(\tau))|^2 d\tau = \sup_{s \in R} \int_{s-t}^{s+t} |g(y_t(\tau))|^2 d\tau = m_t^2 .$$

Take $T > 0$ (we will later choose T sufficiently large) and let $t > T$. In the estimates which follow we repeatedly obtain upper bounds f_i ; which are functions of T. Each function $f_i(T)$ is a priori given by g, α and K. In particular note that each f_i is independent of t and y_t. An odd-indexed bound $f_{2n+1}(T)$ is always a monotonically decreasing function of T and satisfies

$$\lim_{T\to\infty} f_{2n+1}(T) = 0 ,$$

whereas an even-indexed bound $f_{2n}(T)$ satisfies $f_{2n} \in L^\infty_{loc}(R^+)$.

Multiply (1.26) by $g(y_t(\tau))$, integrate over $[-t,t]$, split the integral term in two parts and define z_t, g_K , G_K by

$$(3.4) \qquad z_t(\tau) = g(y_t(\tau)), \quad |\tau| < t \; ; \quad z_t(\tau) = 0 , \quad |\tau| > t ,$$

$g_K = \sup_{|x|<K} |g(x)|$, $G_K = \sup_{|x|<K} |G(x)|$. This gives, after an application of Parseval's relation,

$$(3.5) \qquad (2\pi)^{-1} \int_R |\hat{z}_t|^2 \mathrm{Re}\, \hat{\alpha}(\omega) d\omega < 2G_K + \Big| \int_{-t}^{t} g(y_t(\tau)) \int_{(\tau+t,\infty)} g(y_t(\tau-s)) d\alpha(s) d\tau \Big| .$$

As α is positive definite one has by (3.3) and after estimating the right side of (3.5) (see Assertion 1 of [4])

$$(3.6) \qquad \int_R |\hat{z}_t \mathrm{Re}\, \hat{\alpha}|^2 d\omega < m_t^2 f_1(T) + f_2(T) \; ; \qquad t > T .$$

Define u_t, f_t by

$$(3.7) \qquad u_t(\tau) = y_t'(\tau) , \quad |\tau| < t \; ; \quad u_t(\tau) = 0 , \quad |\tau| > t ,$$

(3.8)
$$f_t(\tau) = \begin{cases} 0 & \tau < -t \\ -\int_{(\tau+t,\infty)} g(y_t(\tau-s))\,d\alpha(s), & |\tau| < t \\ \int_{[\tau-t,\tau+t]} g(y_t(\tau-s))\,d\alpha(s), & \tau > t\ . \end{cases}$$

Then

(3.9) $$u_t(\tau) + \int_R z_t(\tau-s)\,d\alpha(s) = f_t(\tau) \quad \text{a.e. on } R\ .$$

Note that as α is finite and u_t, z_t have compact support then $u_t, z_t, f_t \in (L^1 \cap L^2)(R)$ and so the Fourier transforms to follow are well-defined.

Choose $\omega_0 \in (0,1)$ such that (recall the second part of (1.23))

(3.10) $$2\ \mathrm{Re}\ \hat{a}(\omega) > \hat{a}(0)\ , \quad |\omega| < \omega_0\ ,$$

and let $\lambda > 0$ satisfy

(3.11) $$|g(x) - g(y)| < \lambda|x-y|\ , \quad \text{for } |x|, |y| < K\ .$$

Denote $\alpha_0 \overset{\text{def}}{=} \max(1, \sup_{\omega\in R} |\hat{\alpha}(\omega)|^2)$, $\beta \overset{\text{def}}{=} \inf_{\omega\in Z_2} \mathrm{Re}\ \hat{a}(\omega)$. Then take any $\varepsilon \in (0,1)$ such that $\varepsilon\omega_0 < 1$ and such that

(3.12) $$\varepsilon < 8^{-1}[\lambda^2\omega_0^{-2} + \lambda^2\alpha_0]^{-1}$$

(3.13) $$2\ \mathrm{Re}\ \hat{a}(\omega) > \beta > 0 \quad \text{for} \quad \omega \in S_0 \quad \text{where}$$

(3.14) $$S_0 \overset{\text{def}}{=} \{\omega | \mathrm{dist}(\omega,Z_2) < \varepsilon\ , |\mathrm{Im}\ \hat{\alpha}|^2 > \varepsilon\ , \omega_0 < |\omega| < \varepsilon^{-1}\}.$$

Divide R in four pairwise disjoint parts S_i as follows:

(3.15) $$S_1 \overset{\text{def}}{=} \{\omega | |\omega| > \varepsilon^{-1}\}$$

(3.16) $$S_2 \overset{\text{def}}{=} \{\omega |\ \omega_0 < |\omega| < \varepsilon^{-1}\ , \ |\mathrm{Im}\ \hat{\alpha}|^2 < \varepsilon\}$$

(3.17) $$S_3 \overset{\text{def}}{=} \{\omega | \omega_0 < |\omega| < \varepsilon^{-1},\ |\mathrm{Im}\ \hat{\alpha}|^2 > \varepsilon,\ \mathrm{dist}(\omega,Z_2) > \varepsilon\}$$

(3.18) $$S_4 \overset{\text{def}}{=} S_0 \cup \{\omega | |\omega| < \omega_0\}\ .$$

Note that $R = \bigcup_{i=1}^{4} S_i$. In what follows $K_i(\varepsilon,T)$ will denote bounds which are independent of t and y_t but do depend on ε and T .

Our next goal is to show that there exists a constant c_1 (depending only on $\omega_0, \lambda, \alpha_0$ and in particular independent of t, y_t, ε, T) such that provided T is fixed sufficiently large then

(3.19) $$\int_{R\setminus S_4} |\hat{u}_t|^2 d\omega < \varepsilon \int_{|\omega|<\omega_0} |\hat{z}_t|^2 d\omega + \varepsilon\, c_1 \int_{S_4} |\hat{u}_t|^2 d\omega + K_1(\varepsilon,T)$$

for $t > T$.

By (3.9)

$$\hat{u}_t(\omega) + \hat{z}_t(\omega)\hat{\alpha}(\omega) = \hat{f}_t(\omega), \quad \omega \in R, \tag{3.20}$$

and so

$$2^{-1}|\hat{u}_t|^2 < |\hat{z}_t\hat{\alpha}|^2 + |\hat{f}_t|^2 . \tag{3.21}$$

Integrate (3.21) over $R\backslash S_4$ and estimate the right side. Obviously

$$\int_{S_1} |\hat{z}_t\hat{\alpha}|^2 d\omega < \alpha_0 \int_{\varepsilon^{-1}<|\omega|} |\hat{z}_t|^2 d\omega \tag{3.22}$$

$$\int_{S_2} |\hat{z}_t\hat{\alpha}|^2 d\omega < \varepsilon \int_{\omega_0<|\omega|} |\hat{z}_t|^2 d\omega + \int_R |\hat{z}_t \operatorname{Re} \hat{\alpha}|^2 d\omega . \tag{3.23}$$

Then note that by (1.22), (1.23), (3.17) there exists $\delta = \delta(\varepsilon) \in (0,1)$ such that $\operatorname{Re} \hat{\alpha}(\omega) > \delta^{1/2}$, $\omega \in S_3$. Take any such δ. Then

$$\int_{S_3} |\hat{z}_t\hat{\alpha}|^2 d\omega < 2\alpha_0\delta^{-1} \int_R |\hat{z}_t \operatorname{Re} \hat{\alpha}|^2 d\omega . \tag{3.24}$$

By slight modifications of the estimates of Assertion 2 of [4] one gets

$$\int_R |\hat{f}_t|^2 d\omega < m_t^2 f_3(T) + f_4(T), \quad t > T . \tag{3.25}$$

From (3.21)-(3.25) and from (3.6) follows

$$2^{-1} \int_{R\backslash S_4} |\hat{u}_t|^2 d\omega < \delta^{-1}m_t^2 f_5(T) + \delta^{-1}f_6(T) + \alpha_0 \int_{\varepsilon^{-1}<|\omega|} |\hat{z}_t|^2 d\omega + \varepsilon \int_{\omega_0<|\omega|} |\hat{z}_t|^2 d\omega . \tag{3.26}$$

By straightforward estimates and making use of (3.11) one gets for any $\gamma > 0$

$$\int_{\gamma<|\omega|} |\hat{z}_t|^2 d\omega < 2\lambda^2\gamma^{-2} \int_R |\hat{u}_t|^2 d\omega + 4g_K^2 \gamma^{-1} . \tag{3.27}$$

Use (3.27) (with $\gamma = \omega_0, \varepsilon^{-1}$) to estimate the right side of (3.26). (Note that $m_t^2 = \int_{\omega_0<|\omega|} |\hat{z}_t|^2 d\omega + \int_{|\omega|<\omega_0} |\hat{z}_t|^2 d\omega$.) This yields

$$\left\{ \begin{aligned} 4^{-1} \int_{R\backslash S_4} |\hat{u}_t|^2 d\omega &< \delta^{-1}f_5(T) \int_{|\omega|<\omega_0} |\hat{z}_t|^2 d\omega + \delta^{-1}f_7(T) \int_R |\hat{u}_t|^2 d\omega \\ &\quad + \varepsilon\, c_0 \int_{S_4} |\hat{u}_t|^2 d\omega + \delta^{-1} f_8(T) \end{aligned} \right. \tag{3.28}$$

where we have also used (3.12) and defined $c_0 = 2\alpha_0\lambda^2 + 2\lambda^2\omega_0^{-2}$. Choose T sufficiently large so that

$$\delta^{-1} f_7(T) < 8^{-1}\varepsilon , \qquad \delta^{-1} f_5(T) < 8^{-1}\varepsilon .$$

From (3.28) one then has, for $t > T$, (recall that $\varepsilon < 1$)

$$\int_{R\setminus S_4} |\hat{u}_t|^2 d\omega < \varepsilon \int_{|\omega|<\omega_0} |\hat{z}_t|^2 d\omega + \varepsilon[1 + 8c_0] \int_{S_4} |\hat{u}_t|^2 d\omega + 8^{-1}\delta f_8(T) , \tag{3.29}$$

and so (3.19) holds, with $c_1 = 1 + 8c_0$ and $K_1 = 8\delta^{-1} f_8$.

In what follows we wish to eliminate the second integral on the right side of (3.29). Thus we show that there exists a constant c_2 (depending only on $\omega_0, \lambda, \alpha_0$) such that provided T is fixed sufficiently large then

$$\int_{R\setminus S_4} |\hat{u}_t|^2 d\omega < \varepsilon c_2 \int_{S_4} |\hat{z}_t|^2 d\omega + K_2(\varepsilon,T), \qquad t > T . \tag{3.30}$$

By (3.20), (3.25), provided T is taken so that $f_3(T) < \alpha_0$,

$$\begin{cases} 2^{-1} \int_{S_4} |\hat{u}_t|^2 d\omega < \int_{S_4} |\hat{z}_t \hat{\alpha}|^2 d\omega + \int_{S_4} |\hat{f}_t|^2 d\omega < \\[2ex] 2\alpha_0 \int_{S_4} |\hat{z}_t|^2 d\omega + f_3(T) \int_{R\setminus S_4} |\hat{z}_t|^2 d\omega + f_4(T) . \end{cases} \tag{3.31}$$

Invoke (3.27) with $\gamma = \omega_0$ and then (3.19) to obtain

$$\int_{R\setminus S_4} |\hat{z}_t|^2 d\omega < \int_{\omega_0<|\omega|} |\hat{z}_t|^2 d\omega < 2\lambda^2\omega_0^{-2}\varepsilon \int_{|\omega|<\omega_0} |\hat{z}_t|^2 d\omega + \tilde{c}_1 \int_{S_4} |\hat{u}_t|^2 d\omega + \tilde{K}_2(\varepsilon,T) \tag{3.32}$$

where $\tilde{c}_1 = 2\lambda^2\omega_0^{-2}[1+c_1]$; $\tilde{K}_2 = 2\lambda^2\omega_0^{-2}K_1 + 4q_K^2\omega_0^{-1}$. Now use (3.32) to estimate the last integral on the right side of (3.31). This yields

$$\int_{S_4} |\hat{u}_t|^2 d\omega < 12\alpha_0 \int_{S_4} |\hat{z}_t|^2 d\omega + 4f_4(T) + 4f_3(T)\tilde{K}_2(\varepsilon,T), \qquad t > T , \tag{3.33}$$

provided T is taken such that $f_3(T)\tilde{c}_1 < 4^{-1}$; $f_3(T)2\lambda^2\omega_0^{-2}\varepsilon < \alpha_0$. Finally estimate the right side of (3.19) with the aid of (3.33). The relation (3.30) follows, with $c_2 = 1 + 12\alpha_0 c_1$.

Take $\gamma = \omega_0$ in (3.27), add $\int_{|\omega|<\omega_0} |\hat{z}_t|^2 d\omega$ to both sides and use (3.30), (3.33) to estimate the right side. One obtains

$$\int_{R} |\hat{z}_t|^2 d\omega < \tilde{c}_2 \int_{S_4} |\hat{z}_t|^2 d\omega + K_3(\varepsilon,T), \qquad t > T , \tag{3.34}$$

where $\tilde{c}_2 = 1 + 2\lambda^2\omega_0^{-2}[c_2 + 12\alpha_0]$. Use (3.34) in (3.25) to get

(3.35) $$\int_R |\hat{f}_t|^2 d\omega < \tilde{c}_2 f_3(T) \int_{S_4} |\hat{z}_t|^2 d\omega + K_4(\varepsilon,T), \qquad t > T,$$

where $K_4 = f_3K_3 + f_4$.

By (3.20) $|\hat{z}_t\hat{\alpha}|^2 < 2|\hat{u}_t|^2 + 2|\hat{f}_t|^2$. Integrate this inequality over $R\backslash S_4$ and invoke (3.30), (3.35). This yields

(3.36) $$\int_{R\backslash S_4} |\hat{z}_t\hat{\alpha}|^2 d\omega < 2\varepsilon[c_2 + \tilde{c}_2] \int_{S_4} |\hat{z}_t|^2 d\omega + K_5(\varepsilon,T), \qquad t > T,$$

provided T is taken such that $f_3(T) < \varepsilon$. But $|\hat{\alpha}| = |\omega\hat{a}|$ and hence

(3.37) $$\int_{R\backslash S_4} |\hat{z}_t \text{ Re } \hat{a}|^2 d\omega < \omega_0^{-2} \int_{R\backslash S_4} |\hat{z}_t\hat{\alpha}|^2 d\omega$$

which together with (3.36) implies

(3.38) $$\int_{R\backslash S_4} |\hat{z}_t \text{ Re } \hat{a}|^2 d\omega < \varepsilon c_3^2 \int_{S_4} |\hat{z}_t|^2 d\omega + \omega_0^{-2} K_5(\varepsilon,T),$$

for $t > T$ and where $c_3^2 = 2\omega_0^{-2}[c_2 + \tilde{c}_2]$.

Define A_ε, B_ε by

(3.39) $$A_\varepsilon = \{\omega | \omega \in R\backslash S_4, \quad |\text{Re } \hat{a}(\omega)| \geq c_3\varepsilon^{1/2}\}$$

(3.40) $$B_\varepsilon = \{\omega | \omega \in R\backslash S_4, \quad |\text{Re } \hat{a}(\omega)| < c_3\varepsilon^{1/2}\}.$$

A combination of (3.38), (3.39) results in

(3.41) $$\left|\int_{A_\varepsilon} \text{Re } \hat{a}|\hat{z}_t|^2 d\omega\right| < c_3^{-1}\varepsilon^{-1/2} \int_{A_\varepsilon} |\hat{z}_t \text{ Re } \hat{a}|^2 d\omega < c_3\varepsilon^{1/2} \int_{S_4} |\hat{z}_t|^2 d\omega + K_6(\varepsilon,T)$$

and from (3.34), (3.40) follows

(3.42) $$\left|\int_{B_\varepsilon} \text{Re } \hat{a}|\hat{z}_t|^2 d\omega\right| < c_3\varepsilon^{1/2} \int_{R\backslash S_4} |\hat{z}_t|^2 d\omega < c_3\tilde{c}_2\varepsilon^{1/2} \int_{S_4} |\hat{z}_t|^2 d\omega + K_7(\varepsilon,T).$$

Consequently

(3.43) $$\left|\int_{R\backslash S_4} \text{Re } \hat{a}|\hat{z}_t|^2 d\omega\right| < c_4\varepsilon^{1/2} \int_{S_4} |\hat{z}_t|^2 d\omega + K_8(\varepsilon,T), \qquad t > T,$$

where $c_4 = c_3(1 + \tilde{c}_2)$.

Multiply (1.18) by z_t, integrate over $[-t,t]$ and use Parseval's relation. This gives

$$(3.44)\quad \int_{-t}^{t} z_t(\tau)y_t(\tau)d\tau + \int_R |\hat{z}_t(\omega)|^2 \mathrm{Re}\,\hat{a}(\omega)d\omega = -\int_{-t}^{t} z_t(\tau)\int_{(\tau+t,\infty)} g(y_t(\tau-s))a(s)ds d\tau .$$

The right side of (3.44) ($\overset{\text{def}}{=} r(t)$) can be shown to satisfy (compare (3.5), (3.6) and use (3.34))

$$(3.45)\qquad |r(t)| \le f_5(T)\int_{S_4} |\hat{z}_t|^2 d\omega + K_9(\varepsilon,T) , \qquad t > T .$$

Combine (3.43)-(3.45), use $yg(y) \geq 0$, $|y| \le K$ (note that this is the only place where this condition is used) and recall that by (3.10), (3.13), (3.18) $2\,\mathrm{Re}\,\hat{a}(\omega) \geq \beta > 0$, $\omega \in S_4$. This yields

$$(3.46)\qquad \frac{\beta}{2}\int_{S_4} |\hat{z}_t|^2 d\omega \le \varepsilon^{1/2}[c_4+1]\int_{S_4} |\hat{z}_t|^2 d\omega + K_{10}(\varepsilon,T), \qquad t > T ,$$

provided T is taken such that $f_5(T) \le \varepsilon^{1/2}$. But $\lim_{t\to\infty} m_t^2 = \infty$ and so, by (3.34), we have $\lim_{t\to\infty} \int_{S_4} |\hat{z}_t|^2 d\omega = \infty$. An examination of (3.46) reveals that this implies

$\beta \le 2\varepsilon^{1/2}[c_4 + 1]$. The constants c_4 and β are however independent of ε, which was taken sufficiently small but otherwise arbitrary and hence a contradiction follows.

We conclude that $\sup_{t>0} m_t^2 < \infty$ which gives the first part of (1.25). The second part is a consquence of (1.26), of the first part and of the fact that α is finite. The proof of Theorem 2 is complete.

4. PROOF OF THEOREM 3 .

We begin by proving the following

LEMMA 3. *Let the assumptions of Theorem* 2 *hold and let* y *be a nonconstant solution of* (1.18). *Then* $(G(x) \overset{\text{def}}{=} \int_0^x g(u)du)$

$$\lim_{t\to\infty} G(y(t)) < \lim_{t\to-\infty} G(y(t)). \tag{4.1}$$

Proof of Lemma 3. By (1.25) both limits in (4.1) exist. Multiply (1.26) by $g(y(t))$ and integrate over R . This gives - by (1.20), (1.25) the integral is well defined -

$$G(y(\infty)) - G(y(-\infty)) + (2\pi)^{-1} \int_R |\hat{g}(\omega)|^2 \operatorname{Re} \hat{\alpha}(\omega)d\omega = 0 , \tag{4.2}$$

where $\hat{g}$ is the Fourier transform of $g(y(t))$. But as $\operatorname{Re} \hat{\alpha} > 0$ and as $m(\{\omega|\operatorname{Re} \hat{\alpha}(\omega) = 0\}) = 0$ we have (4.1) provided $m(\{\omega|\hat{g}(\omega) \neq 0\}) > 0$. However, as $y(t) \not\equiv$ constant we have $y'(t) \not\equiv 0$ and so by (1.26) $\int_{R^+} g(y(t-s))d\alpha(s) \not\equiv 0$. As $\alpha(R^+) = 0$ this implies $g(y(t)) \not\equiv$ constant and consequently $\hat{g}$ differs from zero on a set of nonzero measure. Lemma 3 is proved.

With this Lemma and (1.25) one can easily repeat the arguments of [9, Lemma 5.1a and final part of the proof of Theorem 1a] to obtain (1.31). (Note that although we now do have $\alpha(R^+) = 0$ we nevertheless do not have to resort to energy functions of type $G(x) + 2^{-1}g^2(x) \int_{R^+} a(s)ds$).

To get (1.32) it suffices to integrate (1.27) and to recall (1.21), (1.29) and (1.31).

5. PROOF OF THEOREM 4.

The first part of the proof closely follows that of Theorem 1.

Convolve (1.1) with r_λ, $\lambda \in (0,\delta)$, and use (1.36). Perform an integration by parts and define $h_\lambda(x) = \lambda^{-1}g(x) - x$, $x \in R$; $c_\lambda(t) = -r'_\lambda(t)$, $t \in R^+$. This gives, after differentiating

(5.1) $$x'(t) + \int_{[0,t]} h_\lambda(x(t-s))dc_\lambda(s) = z'_\lambda(t)\ , \quad \text{a.e. on } R^+ .$$

The relations (2.4)-(2.6) hold with μ replaced with $\lambda\mu$ and c with c_λ. Hence $(\hat{c}_\lambda \overset{\text{def}}{=} \int_{R^+} e^{-i\omega t}dc_\lambda(t))$,

(5.2) $$\text{Re } \hat{c}_\lambda(\omega) \geq 0\ , \quad \omega \in R, \quad \lambda > 0\ ,$$

with

(5.3) $$\text{Re } \hat{c}_\lambda(\omega) = 0\ , \quad \text{iff} \quad \omega \in Z_1 \cup Z_2 \cup \{0\}\ ,$$

where

(5.4) $$Z_1 \overset{\text{def}}{=} \{\omega|\ \omega \neq 0,\ \text{Re } \hat{\mu}(\omega) = 0\ ,\ |\hat{\mu}| < \infty\}\ ,\quad Z_2 \overset{\text{def}}{=} \{\omega|,\ \omega \neq 0,\ |\hat{\mu}(\omega)| = \infty\}.$$

By by (1.33)

(5.5) $$\text{Im } \hat{c}_\lambda(\omega) = 0\ , \quad \omega \in Z_1\ .$$

Then note that

(5.6) $$\int_{R^+} e^{-i\omega t}c_\lambda(t)dt = 1\ , \quad \omega \in Z_2\ ,$$

(5.7) $$\int_{R^+} e^{-i\omega t}c_\lambda(t)dt = 1 - r_\lambda(\infty) > 0, \quad \omega = 0\ ,$$

where (5.7) is a consequence of (1.37), (1.39) and Lemma 2. From (1.19) follows

(5.8) $$h_\lambda(x) \text{ is locally Lipschitzian}, \quad x \in R, \quad \lambda > 0\ ,$$

and invoking (1.34), (1.35) one has

(5.9) $$xh_\lambda(x) \geq 0\ , \quad \text{for} \quad |x| \leq |x(t)|_{L^\infty(R^+)}\ ,$$

provided λ is taken sufficiently small.

By (1.37), (1.38), (1.40), (5.2)-(5.9) an application of Theorem 3 to (5.1) is permitted. The relation (1.41) and

(5.10) $$\lim_{t\to\infty} \{[1-r_\lambda(\infty)]g(x(t)) + \lambda r_\lambda(\infty)x(t)\} = \lambda z_\lambda(\infty)$$

follows.

Our final goal is to obtain (1.42), (1.43) from (5.10). By Lemma 2 and (1.37), (1.39) we have $\gamma = \lambda r_\lambda(\infty)[1-r_\lambda(\infty)]^{-1}$. Suppose at first that $\gamma > 0$ (thus $r_\lambda(\infty) \in (0,1)$) and recall that $z_\lambda = x_0 r_\lambda + w_\lambda$, where $w_\lambda = r_\lambda * f$. Obviously

(5.11) $$\lim_{t\to\infty} \{\lambda[1-r_\lambda(\infty)]^{-1} x_0 r_\lambda(t)\} = x_0\gamma .$$

Then note that as $w'_\lambda = f + r'_\lambda * f$ the Laplace transform of $w'_\lambda(t)$ is well defined for $\operatorname{Re} s > 0$. But as $\int_0^t w'_\lambda(\tau)d\tau = w_\lambda(\infty)$ exists it follows that [12, p. 183] (s real)

(5.12) $$\begin{cases} w_\lambda(\infty) = \lim\limits_{s\to 0+} [\tilde{f}(s) + \tilde{r}'_\lambda(s)\tilde{f}(s)] = \lim\limits_{s\to 0+} s\tilde{f}(s)[s + \lambda\tilde{\mu}(s)]^{-1} \\[2ex] = \lim\limits_{s\to 0+} \left[\tilde{f}(s) \dfrac{s[\lambda\tilde{\mu}(s)]^{-1}}{s[\lambda\tilde{\mu}(s)]^{-1}+1}\right] = r_\lambda(\infty) \lim\limits_{s\to 0+} \tilde{f}(s) \end{cases}$$

where we have used the fact that as γ is finite there exists $\varepsilon > 0$ such that $\tilde{\mu}(s) \neq 0$ for $s \in \{s | \operatorname{Re} s > 0 ,\ 0 < |s| < \varepsilon\}$ and the assumption $r_\lambda(\infty) > 0$. Note that the existence of $\lim\limits_{s\to 0+} \tilde{f}(s)$ is part of the result of (5.12). Hence,

(5.13) $$\lambda[1 - r_\lambda(\infty)]^{-1} w_\lambda(\infty) = \gamma \lim_{s\to 0+} \tilde{f}(s) ,$$

and so by (5.10), (5.11), (5.13), in case $\gamma > 0$,

(5.14) $$\lim_{t\to\infty} \{g(x(t)) + \gamma x(t)\} = \gamma[x_0 + \lim_{s\to 0+} \tilde{f}(s)] .$$

Suppose next that

(5.15) $$\gamma = 0 .$$

Then $r_\lambda(\infty) = 0$ and so

(5.16) $$\lim_{t\to\infty} x_0 r_\lambda(t) = 0 .$$

Because the existence of $\lim\limits_{t\to\infty} (r_\lambda * f)(t)$ is assumed we get

(5.17) $$\begin{cases} \lim\limits_{t\to\infty} \lambda(r_\lambda * f)(t) = \lambda \lim\limits_{s\to 0+} \{s\tilde{f}[s + \lambda\tilde{\mu}]^{-1}\} = \\[2ex] \lambda \lim\limits_{s\to 0+} \left\{ \dfrac{s[\tilde{\mu}]^{-1}\tilde{f}}{s[\tilde{\mu}]^{-1}+\lambda} \right\} = \lim\limits_{s\to 0+} \{s[\tilde{\mu}]^{-1}\tilde{f}\} , \end{cases}$$

as $\lim_{s \to 0+} s[\tilde{\mu}(s)]^{-1} = 0$ and $\lambda > 0$. Thus, by (5.10), (5.16), (5.17)

$$\lim_{t \to \infty} g(x(t)) = \lim_{s \to 0+} \{s[\tilde{\mu}(s)]^{-1} \tilde{f}(s)\} , \tag{5.18}$$

if (5.15) holds.

6. PROOF OF THEOREM 5.

The method of the previous section enables us to transform (1.1) into

(6.1) $$x'(t) - \int_{[0,t]} h_\lambda(x(t-s))dr'_\lambda(s) = z'_\lambda(t) \;, \quad \text{a.e. on } R^+ ,$$

where $h_\lambda(x) \overset{def}{=} \lambda^{-1}g(x) - x$. Multiply (6.1) by $h_\lambda(x(t))$, integrate with respect to t over $[0.T]$ and use the fact that $-dr'_\lambda$ generates a positive definite measure. This yields

(6.2) $$H_\lambda(x(T)) - H_\lambda(x(0)) \leq \int_0^T h_\lambda(x(t))z'_\lambda(t)dt \;,$$

where $H_\lambda(x) = \lambda^{-1}G(x) - 2^{-1}x^2$. Making use of (1.50) one shows that for any sufficiently small λ there exist constants c_1, c_2 (depending on λ) such that $|h_\lambda(x)| \leq c_1 + c_2 H_\lambda(x)$, $x \in R$. Therefore, by a simple application of Gronwall's inequality to (6.2) and recalling (1.51) we get

(6.3) $$\sup_{T>0} H_\lambda(x(T)) < \infty \,.$$

But (6.3), the second part of (1.50), and the definition of H_λ imply

(6.4) $$\sup_{t\in R^+} |x(t)| < \infty \,.$$

We finally point out that under the present assumptions one also has, for all sufficiently small λ,

$$\sup_{T>0} Q(h_\lambda(x(t)),-dr',T) < \infty \,,$$

from which various consequences concerning the asymptotic behavior of $x(t)$ can be deduced.

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Let $g \in C(R)$, $f \in L^1_{loc}(R^+)$ and let μ be a real locally finite positive definite Borel measure on R^+. We investigate a relation between the solutions of the nonlinear scalar Volterra equation

$$x'(t) + \int_{[0,t]} g(x(t-s))d\mu(s) = f(t), \quad t \in R^+, \quad x(0) = x_0 ,$$

20. Abstract (Continued)

and the solutions of linear equations with the same data

$$z_\lambda'(t) + \lambda \int_{[0,t]} z_\lambda(t-s)d\mu(s) = f(t), \quad t \in R^+, \quad z_\lambda(0) = x_0 , \quad \lambda > 0 .$$

This relation, when combined with results (established in this paper) on the global size of solutions of certain limit equations

$$y(t) + \int_{R^+} g(y(t-s))a(s)ds = 0 , \qquad t \in R ,$$

allows us to obtain new asymptotic results for the solution $x(t)$ in the case when both μ and f are large in a precise sense.

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